Team Play Topics

Round Two

The first section of the round two Mandelbrot Team Play is reproduced below. A list of practice problems is also provided to aid in preparation. Note that these problems are not meant so serve as a precise indicator of the problems that will appear on the contest. However, students who understand how to solve them should be able to make significantly more progress than they might have otherwise. So work hard on the problems, and good luck on the Team Play!

Definitions: Suppose that a finite number of points are selected around the circumference of a circle. We say that these points are in *general position* if it is not possible to choose six of these points and connect them in pairs with three concurrent segments, i.e. segments that all meet in a single point. For example, sets of five or fewer points are automatically in general position, but eight evenly spaced points around a circle are not in general position.

Practice Problems

1. Suppose that eight points A, B, C, D, E, F, G, and H in general position are given in this order around the circumference of a circle. Draw segments $\overline{AE}$, $\overline{BF}$, $\overline{CG}$, and $\overline{DH}$. How many points of intersection are created by these four segments?

2. If the points in the previous problem are not necessarily in general position, then there could be several possible answers to the question. What numbers of points of intersection are possible?

3. For $n \geq 1$ select $2n$ points on the circumference of a circle. Let $G(n)$ tally the number of different ways to connect all these points in pairs with n segments so that none of the segments intersect. Compute $G(1)$, $G(2)$, and $G(3)$.

4. Now compute $G(4)$. (Hint: some of the valid configurations can be obtained from others by rotations, which will cut down on the casework.)

5. It will be convenient to define $G(0) = 1$. Prove that for $n \geq 1$ we have

$$G(n) = G(n-1)G(0) + G(n-2)G(1) + \cdots + G(0)G(n-1).$$

Use this recursion to confirm your answer for $G(4)$.

6. (Challenge) For $n \geq 1$, let $H(n)$ count the number of ways to triangulate a regular $(n+2)$-gon. (In other words, draw a regular polygon with $n+2$ sides and count the number of different ways to subdivide it into triangles using nonintersecting diagonals.

Don't peek yet! Solutions on the next page. $\Longrightarrow$

Round Two

1. You should find six points of intersection; one for each pair of segments.

2. If some of the segments are allowed to be concurrent there could be either six, four, or one point of intersection.

3. One finds that $G(1) = 1$, $G(2) = 2$, and $G(3) = 5$.

4. There are a total of $G(4) = 14$ ways to join eight points with nonintersecting segments. However, these arise from only three different configurations: connect adjacent pairs of points (2 ways), connect points with "parallel" segments (4 ways), and two adjacent, two parallel segments (8 ways).

5. Label one of the points as P. If we connect P to a neighboring point, then there are $2n-2$ points left, which can be joined up in $G(n-1)$ ways. There are no valid configurations if we connect P to the next further point around, but if we skip two points then there is $G(1) = 1$ way to connect the two points between, and $G(n-2)$ ways to draw segments among the remaining $2n-4$ points. Continuing around in this manner gives the desired expression. Sure enough,

$$G(4) = G(3)G(0) + G(2)G(1) + G(1)G(2) + G(0)G(3) = 5 + 2 + 2 + 5 = 14.$$

6. One possible strategy is to show that $G(n)$ and $H(n)$ agree for $n = 1$, 2, and 3. Then show that $H(n)$ satisfies the same recursion that $G(n)$ does, namely

$$H(n) = H(n-1)H(0) + H(n-2)H(1) + \cdots + H(0)H(n-1),$$

where we set $H(0) = 1$. This will force all later values of $G(n)$ and $H(n)$ to also agree.

Important note: some students may have noticed that the numbers 1, 2, 5, 14, ... appearing in these problems are the Catalan numbers. However, the round two Team Play is *not* about the sequence of Catalan numbers. As worthwhile an activity as it may be, please do not feel compelled to become experts on Catalan numbers for this contest!