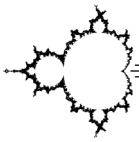
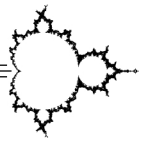


Team Play Topics

Round Three



The first section of the round three Mandelbrot Team Play is reproduced below. A list of practice problems is also provided to aid in preparation. Note that these problems are not meant so serve as a precise indicator of the problems that will appear on the contest. However, students who understand how to solve them should be able to make significantly more progress than they might have otherwise. So work hard on the problems, and good luck on the Team Play!

Facts: Recall that the cosine function takes an angle θ and returns a real number in the range $-1 \leq \cos\theta \leq 1$. Special values include $\cos 0^\circ = 1$, $\cos 90^\circ = 0$, and $\cos 180^\circ = -1$. Also note that $\cos(180^\circ - \theta) = -\cos\theta$. The Law of Cosines states that if a triangle has sides of length a, b, and c with an angle θ opposite the side of length c, then $c^2 = a^2 + b^2 - 2ab\cos\theta$.

Brahmagupta's formula states that a quadrilateral with sides of length a, b, c, and d has area $\sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\phi}$, where $s = \frac{1}{2}(a+b+c+d)$ is the semiperimeter and ϕ is the average of a pair of opposite angles. All the vertices of a *cyclic* quadrilateral lie on a single circle; in this case both pairs of opposite angles are supplementary.

Practice Problems

1. Let ABC be an isosceles triangle with with $AB = AC$. Also let P be any point on side BC and draw segment $\overline{AP}$. We label $AB = AC = a$, $PA = d$, $BP = m$, $PC = n$, and $m\angle APC = \theta$. Write down two different relationships among these quantities using the Law of Cosines, keeping in mind that $\cos(180^\circ - \theta) = -\cos\theta$.

2. Next combine the two equations found in the previous part in a strategic manner and simplify the result in order to conclude that $\cos\theta = (n-m)/(2d)$.

3. Figure out what a quadrilateral must look like if all of its side lengths are 6 and the sum of one pair of opposite angles is 120°. Use basic geometry to determine the area of this quadrilateral. Now apply Brahmagupta's formula to determine its area. (Hopefully your answers agree!)

4. A quadrilateral with sides of length 6, 7, 8, and 11 has area 60. Determine the value of ϕ in Brahmagupta's formula, and hence the sum of a pair of opposite angles. What can you conclude about this quadrilateral?

5. It turns out that Brahmagupta's formula still applies to nonconvex quadrilaterals, and even to self-intersecting quadrilaterals! (In the latter case you have to figure out the correct way to define the area and the angles—some angles will wrap around the "outside," and the smaller triangular piece will contribute a negative amount to the area.) Investigate with several examples of your own to confirm this fact, if you are intersted.

Don't peek yet! Solutions on the next page. $\Longrightarrow$

Round Three

1. Applying the Law of Cosines to triangle APC gives $a^2 = d^2+n^2-2dn\cos\theta$. On the other hand, in triangle ABB we have $a^2 = d^2+m^2-2dm\cos(180^\circ-\theta)$, which becomes $a^2 = d^2+m^2+2dm\cos\theta$, using $\cos(180^\circ-\theta) = -\cos\theta$.

2. If we subtract the first equality from the second above, we are left with

$$0 = m^2 - n^2 + (2dm + 2dn)\cos\theta \qquad \Longrightarrow \qquad n^2 - m^2 = 2d(m+n)\cos\theta.$$

Factoring $n^2 - m^2$ as $(n+m)(n-m)$ and then cancelling the factor of $(m+n)$, which is always positive, we are left with $n - m = 2d\cos\theta$. This yields the desired equality immediately.

3. The quadrilateral must be a rhombus with sides of length 6, two 60° angles, and two 120° angles. Viewing this quadrilateral as two regions which are equilateral triangles, we easily find the overall area to be $18\sqrt{3}$. On the other hand, if we use $a = b = c = d = 6$ (so that $s = 12$) and $\phi = 60^\circ$ (so that $\cos\phi = -\frac{1}{2}$) in Brahmagupta's formula, then we obtain the same area:

$$\textit{area} = \sqrt{(6)(6)(6)(6) - (6)(6)(6)(6)\cdot\tfrac{1}{4}} = 36\sqrt{1-\tfrac{1}{4}} = 18\sqrt{3}.$$

4. We quickly find that $s = 16$. Hence according to Brahmagupta we have

$$60 = \sqrt{(10)(9)(8)(5) - (6)(7)(8)(11)\cos^2\phi} \qquad \Longrightarrow \qquad 3600 = 3600 - 3696\cos^2\phi.$$

Hence $\cos\phi = 0$, or $\phi = 90^\circ$, so a pair of opposite angles adds to 180° in this quadrilateral, which means that we must have a cyclic quadrilateral.

5. Happy exploring!