Fun with polynomials

1. Find the largest solution of $x^3 - 27x^2 + 242x - 720 = 0$ given that one root equals the average of the other two roots. (Mu Alpha Theta 1990)

2. Find the solutions to $x^4 - 7x^3 + 14x^2 - 7x + 1 = 0$ (preferably without using the quartic formula).

3. If a, b, c, d are the solutions of the equation $x^4 - x - 3 = 0$, then find the polynomial with degree 4 and leading coefficient 3 whose roots are

$$\frac{a+b+c}{d^2}, \frac{a+b+d}{c^2}, \frac{a+c+d}{b^2}, \frac{b+c+d}{a^2}.$$

(AHSME 1981)

4. Let $P_0(x) = x^3 + 313x^2 - 77x - 8$. For integers $n \geq 1$, define $P_n(x) = P_{n-1}(x - n)$. What is the coefficient of x in $P_{20}(x)$? (AIME 1993 5)

5. Let C be the coefficient of x^2 in the expansion of the product

$$(1 - x)(1 + 2x)(1 - 3x)...(1 + 14x)(1 - 15x)$$

Find $|C|$. (AIME 2004 I 7)

6. Consider the polynomials $P(x) = x^6 - x^5 - x^3 - x^2 - x$ and $Q(x) = x^4 - x^3 - x^2 - 1$. Given that z_1, z_2, z_3, and z_4 are the roots of $Q(x) = 0$, find $P(z_1) + P(z_2) + P(z_3) + P(z_4)$. (AIME 2003 II 9)

7. The equation $2000x^6 + 100x^5 + 10x^3 + x - 2 = 0$ has exactly two real roots, one of which is $\frac{m+\sqrt{n}}{r}$, where m, n, and r are integers, m and r are relatively prime, and $r > 0$. Find $m + n + r$. (AIME 2000 II 13)